Data-driven reduced-order modelling of dynamo waves

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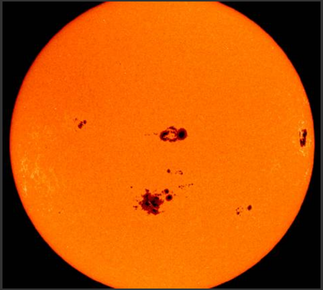
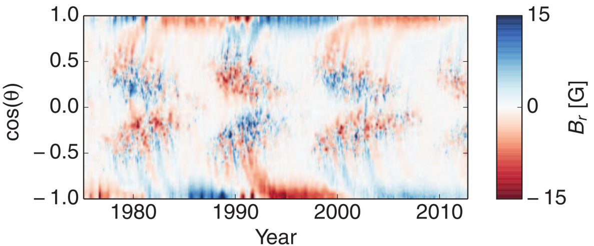
Motivation: astrophysical magnetic fields





Cameron and Schüssler (2015)

How do the large-scale magnetic fields appear and saturate?
Can we describe their dynamics with simple, data-driven models?

1D dynamo models - Bushby (2003)

$$A_t = \underbrace{\cos(\pi x/2) B}_{\alpha\text{-effect}} + K_f A_{xx} + \eta_A A,$$

$$B_t = -\underbrace{D \sin(\pi x/2) A_x}_{\Omega\text{-effect}} + \underbrace{D \left[K_1 u_x A - K_3 u A_x\right]}_{\text{Feedback from } u} + K_g B_{xx} + \eta_B B,$$

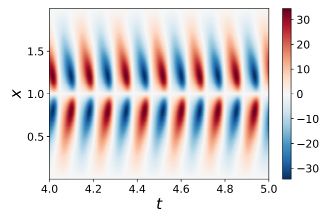
$$u_t = -\underbrace{(K_2 A_x B - K_1 B_x A)}_{\text{“Lorentz” force}} + K_h u_{xx} + \eta_u u$$

$$A = B = u = 0 \text{ at } x = 0, L$$

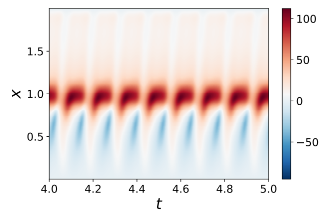
Parameter: dynamo number D

System states: toroidal magnetic field B

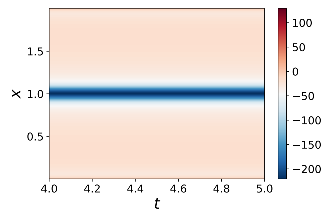
Dipole



Mixed parity



Steady quadrupole

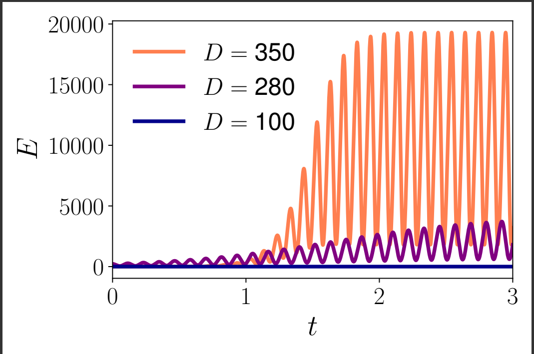


$260 \geq D > 750$ $\quad$ $750 \geq D > 2200$ $\quad$ $D \geq 2200$

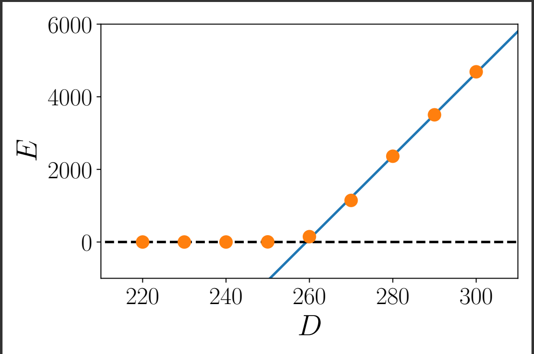
Software: Dedalus

Hopf bifurcation to dipolar state

$B^2 + A_x^2$ a function of time



$B^2 + A_x^2$ as a function of D

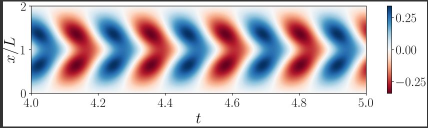


Data-driven approach

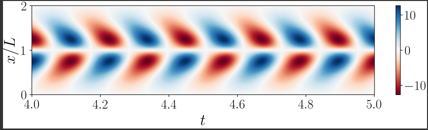
- Extract spatial patterns (modes) from the dynamo data set
  Here: Principal Orthogonal Decomposition (POD)
  POD modes are obtained from SVD of the data matrix: $Q = \Phi\Sigma V^{*}$
- Represent the dynamics of the system with a few spatial modes and their temporal evolution
  E.g. modes containing most of the flow energy
- Construct a low-order model of the dynamo in the new basis, without *a priori* knowledge about underlying equations
  Sparse Identification of Nonlinear Dynamics (SINDy)

POD basis for the dynamo, $D = 300$

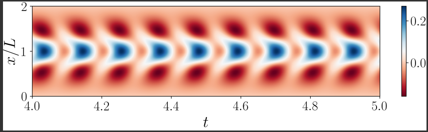
Magnetic potential A

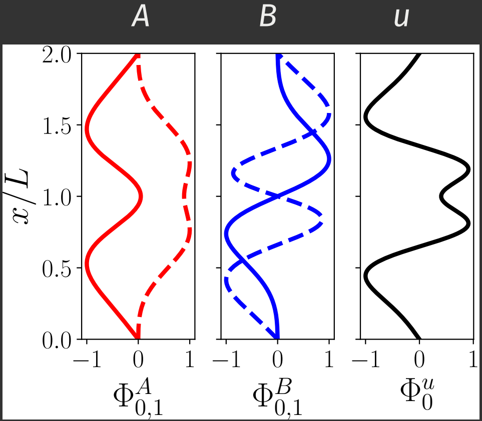


Toroidal field B



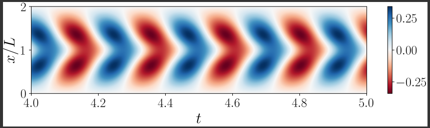
Velocity field u



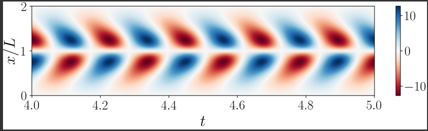


POD basis for the dynamo, $D = 300$

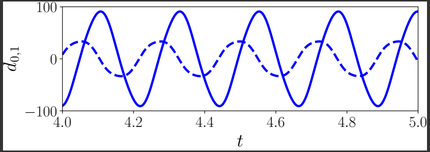
Magnetic potential A

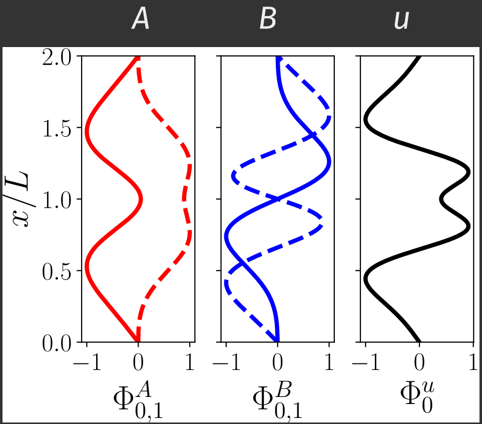


Toroidal field B



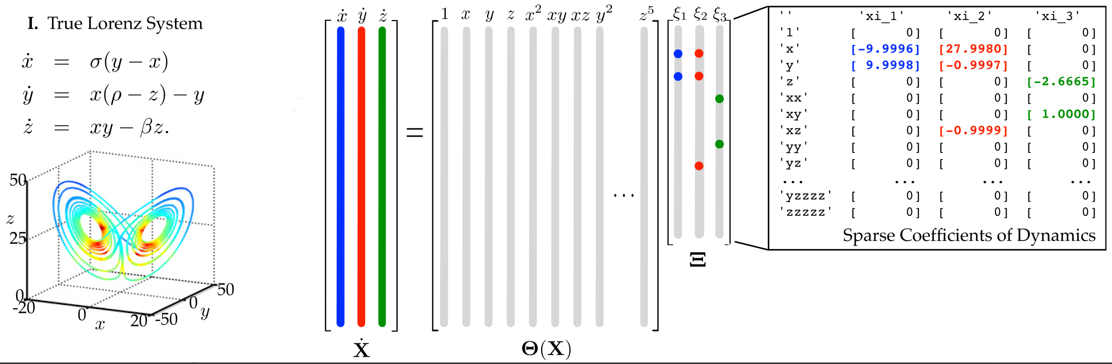
Temporal coefficients of $\Phi^A_{0,1}$, $\Phi^B_{0,1}$





Sparse Identification of Nonlinear Dynamics

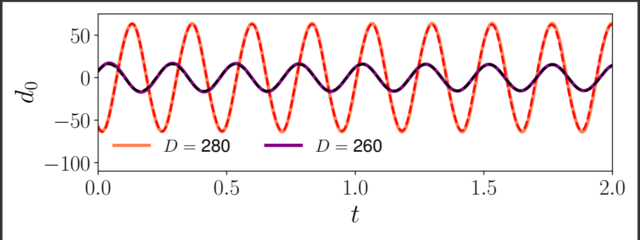
SINDy allows to identify dominant nonlinear dynamics of the system without *a priori* knowledge equations

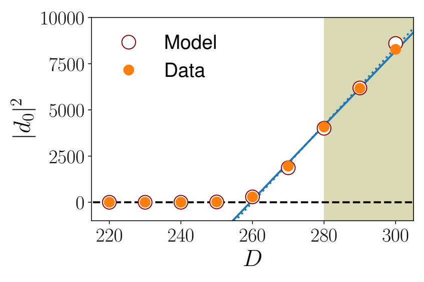


Brunton *et al* (2016), de Silva *et al* (2020) - PySINDy

SINDy model based on the dipole only

$$\dot{d}_0 = -10.310 d_0 + 23.168 d_1 - 0.008 d_0 d_1^2 + D(0.046 d_0 - 0.316 d_1)$$
$$\dot{d}_1 = -0.534 d_0 - 8.118 d_1 - 0.005 d_0 d_1^2 + D(0.045 d_0 + 0.026 d_1)$$



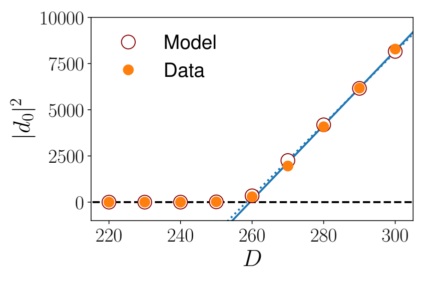


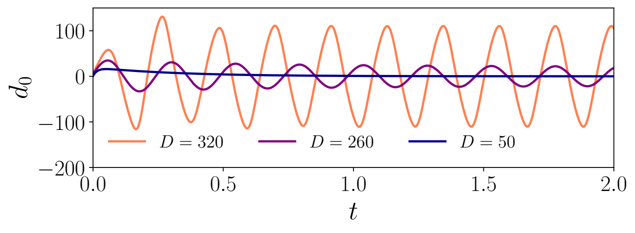
SINDy model based on the dipole and *u*

$$\dot{d}_0 = -14.687d_0 + 80.784d_1 + \underbrace{0.022d_0u_0 - 0.147d_1u_0} + D(0.061d_0 - 0.086d_1)$$

$$\dot{d}_1 = 9.160d_0 - 62.605d_1 - \underbrace{0.039d_0u_0 + 0.051d_1u_0}_{\text{Feedback from } u} - D(0.078d_0 - 0.238d_1)$$

$$\dot{u}_0 = -20.945u_0 - 0.009u_0^2 - \underbrace{0.632d_0^2 + 2.186d_0d_1 + 1.312d_1^2}_{\text{“Lorentz” force}}$$

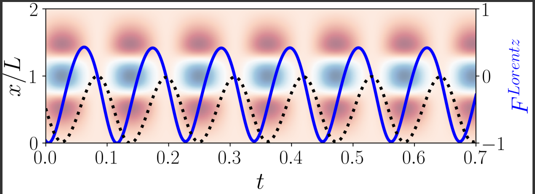




Nonlinear interactions in the model

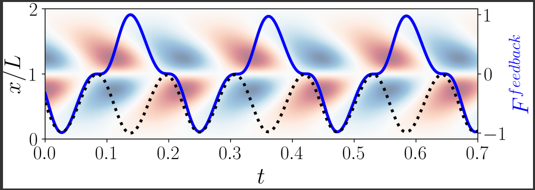
$D = 300$
Dotted line: amplitude of velocity mode u_0

Velocity



"Lorentz" force
$-0.632d_0^2 + 2.186d_0d_1 + 1.312d_1^2$

Toroidal magnetic field B



Feedback from u
$0.022d_0u_0 - 0.147d_1u_0$

Conclusions

We reduced the original PDE model to ODEs that reproduce Hopf bifurcation to dynamo waves qualitatively and quantitatively

PDE system:

$$\begin{aligned} A_t &= \cos(\pi x/2)B + K_f A_{xx} + \eta_A A, \\ B_t &= -D\sin(\pi x/2)A_x + D\left[K_1 u_x A - K_3 u A_x\right] \\ &\quad + K_g B_{xx} + \eta_B B, \\ u_t &= -(K_2 A_x B - K_1 B_x A) + \tau K_h u_{xx} + \tau \eta_u u \end{aligned}$$

⇒

ODE system:

$$\begin{aligned} \dot{d}_0 &= -15d_0 + 81d_1 + 0.02d_0u_0 - 0.2d_1u_0 \\ &\quad + D(0.06d_0 - 0.09d_1) \\ \dot{d}_1 &= 9d_0 - 63d_1 - 0.04d_0u_0 + 0.05d_1u_0 \\ &\quad - D(0.08d_0 - 0.2d_1) \\ \dot{u}_0 &= -21u_0 - 0.009u_0^2 - 0.6d_0^2 + 2d_0d_1 + d_1^2 \end{aligned}$$

SINDy model shows which nonlinear interactions are dynamically important

This data-driven framework can be applied for the dynamo flows far from the instability threshold

Thank you for your attention!

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