INSTABILITIES OF ELEVATOR MODES IN OSCILLATORY DOUBLE-DIFFUSIVE CONVECTION

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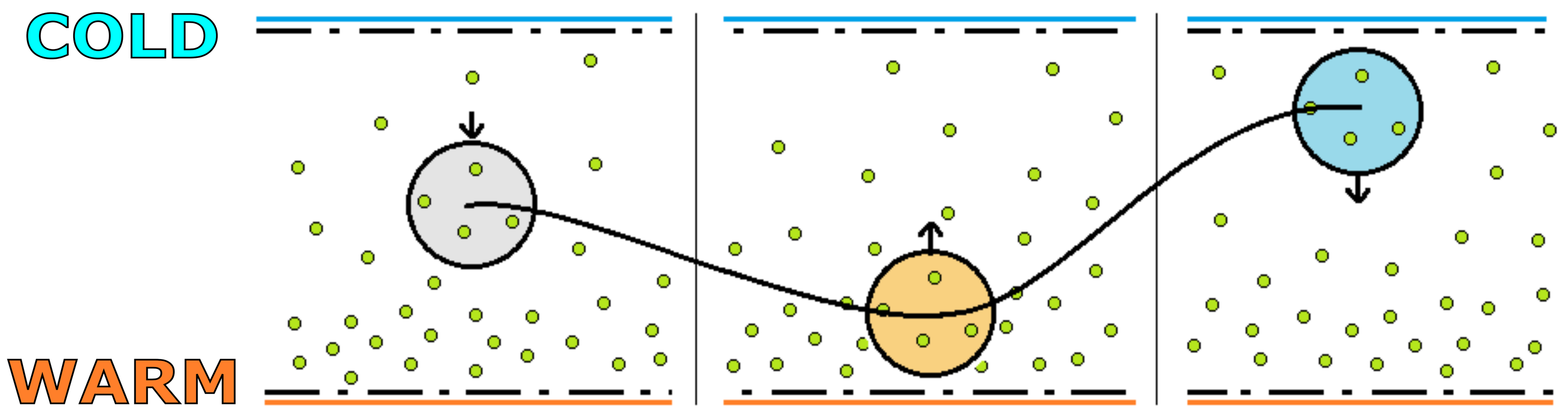
INTRODUCTION

Double-diffusive convection (DDC) arises when two competing elements, such as heat and salt, drive fluid motion through bouyancy effects. Two configurations exist: 'diffusive convection' is seen in the arctic ocean where glacial outflow (cold, fresh) lies over warmer (but still cold!), and saltier seawater; 'salt-fingering' occurs in the tropics where surface evaporation leaves warm, salty water above the cold, fresh outflow from rivers. We look at the former case -- diffusive convection; salt-fingers have been well studied [1,2].



Heat diffuses quicker than salt, so a parcel displaced downwards will rapidly heat up and return to its original position while maintaining its salt content. The parcel may then be warmer than its surroundings, leading to an overshoot and subsequent cooling. This process repeats and may lead to growing oscillations.

The fastest-growing mode (FGM) is a sinusoidal, vertically independent 'elevator' mode which is a non-linear solution to the governing equations -- a solution that grows indefinitely.

Is the nonlinear elevator istelf stable, especially at large amplitudes when it creates large density perturbations and strong shears?

We seek to answer this question in relation to the diffusive regime, building upon previous work for salt-fingers [3,4].

GOVERNING EQUATIONS

In a 2D infinite domain, for an initial density that is a linear function of temperature T, and salt concentration S, i.e.

$$\rho = (-\alpha T + \beta S)$$

where α and β are coefficients of expansion; and with uniform gradients T_z and S_z, the dimensionless governing Boussinesq equations are

$$\left(\frac{\partial}{\partial t} - Pr\nabla^2\right)\nabla^2\psi = -J(\psi,\nabla^2\psi) + Pr\left(\frac{\partial T}{\partial x} - \frac{\partial S}{\partial x}\right),$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T - \frac{\partial\psi}{\partial x} = -J(\psi, T),$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S - R_\rho^{-1}\frac{\partial\psi}{\partial x} = -J(\psi, S),$$

where Pr is the Prandtl number, τ is the ratio of diffusivities, and R_ρ^{-1} is the density gradient ratio, representing the relative strength of background temperature and salinity gradients. In the ocean, we have

$$Pr = \frac{\nu}{\kappa_T} = 7, \qquad \tau = \frac{\kappa_T}{\kappa_S} = 0.01, \qquad 1 < R_\rho^{-1} = \frac{\alpha T_z}{\beta S_z} < 10,$$

where ν is the viscosity, and κ_T and κ_S are the diffusivities of heat and salt respectively.

PRIMARY INSTABILITY: ELEVATOR MODES

The static basic state is unstable to elevators of the form

$$\psi = A_\psi e^{\lambda_0 t}\cos(k_x x),$$

$$T = A_T e^{\lambda_0 t}\sin(k_x x),$$

$$S = A_S e^{\lambda_0 t}\sin(k_x x),$$

where A is the amplitude of perturbations. We solve a cubic equation for the complex growth rate λ_0 to determine regions of instability.

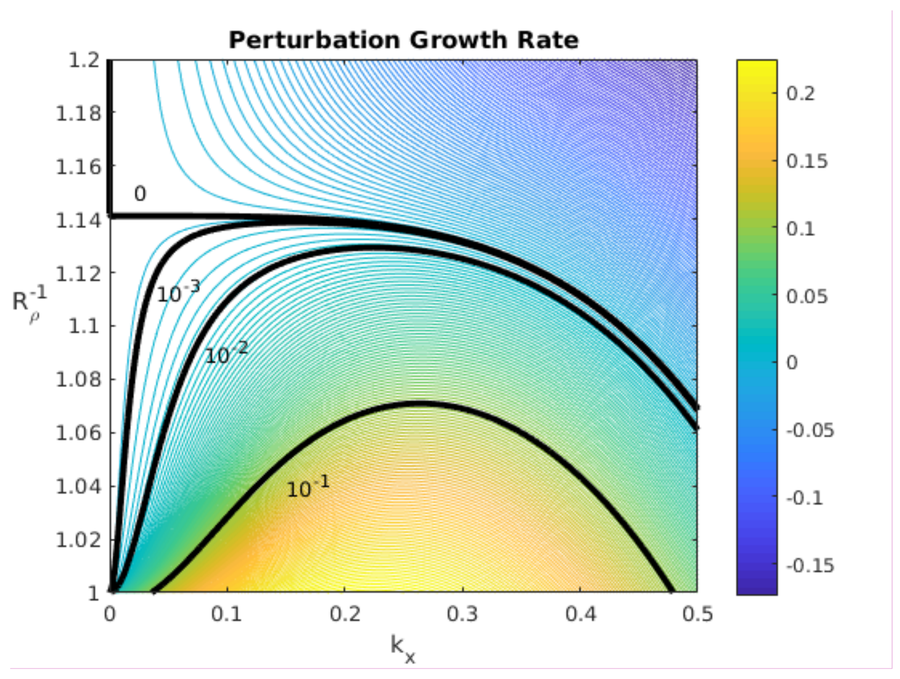


Fig. 1: Growth rates of elevator modes.

The unstable mode always has a non-zero imaginary part, i.e., it is a growing, oscillatory instability; however, marginal, purely oscillatory modes with zero growth rate do exist. Such modes lie on the upper black line in Fig. 1 and are the focus of the present study.

SECONDARY INSTABILITY: FLOQUET EXPANSION

The oscillatory elevator modes introduce periodic coefficients in time and space into the equations governing secondary perturbations. We expand perturbations in the following double-Floquet form:

$$\begin{pmatrix}\psi' \\ T' \\ S'\end{pmatrix} = \exp(ifk_x x + ik_z z + \lambda t)\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\begin{pmatrix}\psi_{n,m} \\ T_{n,m} \\ S_{n,m}\end{pmatrix}\exp(ink_x x + im\omega t),$$

where n and m are integers. Expanding in this way and truncating at $n = [-N, N]$ and $m = [-M, M]$, we obtain an eigenvalue problem of the form

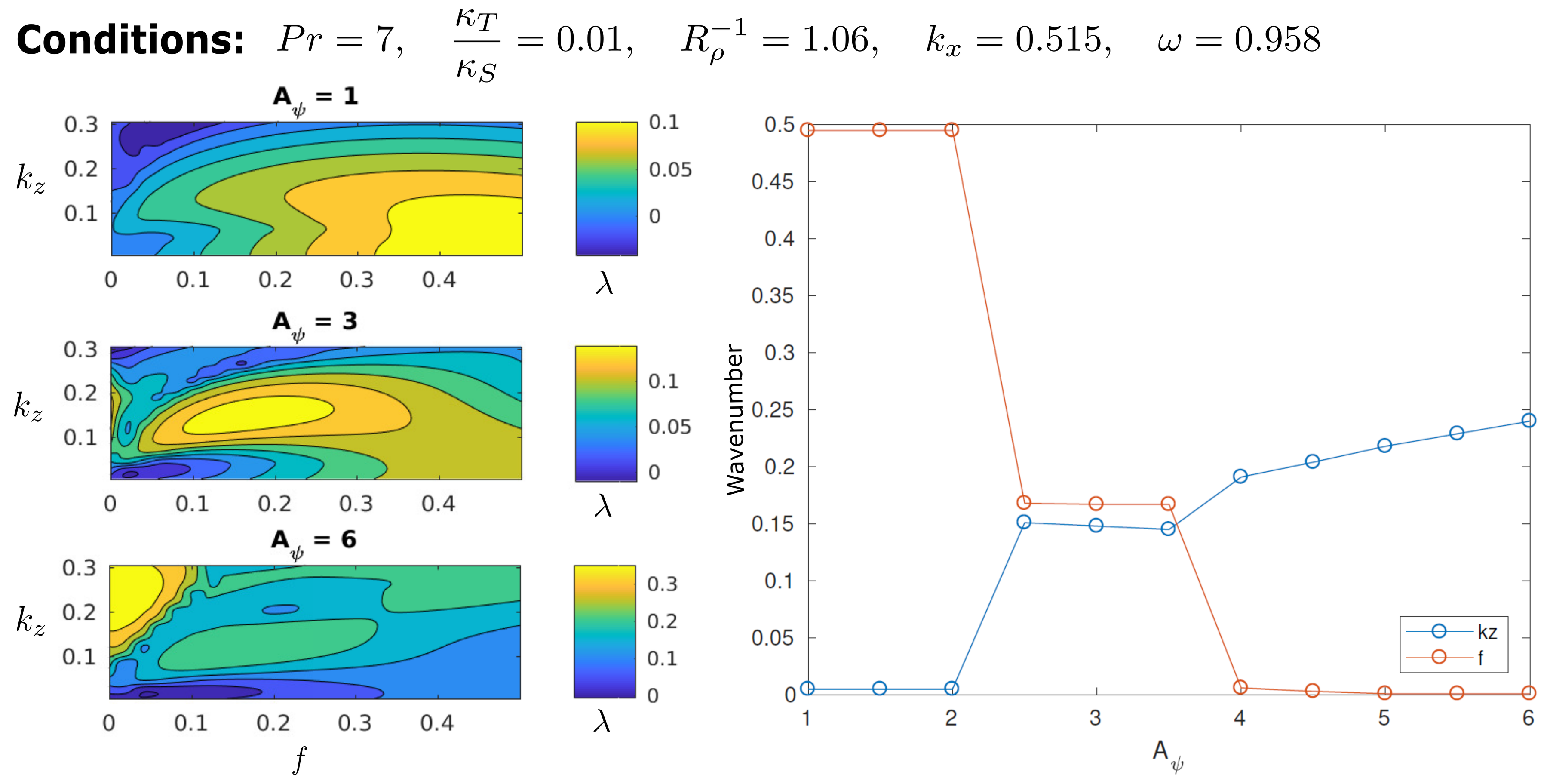
$$A\mu = \lambda\mu,$$

where A is a matrix of size 3(2N+1)(2M+1). We obtain a spectrum of eigenvalues λ with corresponding eigenvectors $(\psi_{n,m}, T_{n,m}, S_{n,m})$, selecting only those with the largest growth rates.

Convergence is generally achieved when $N = M = 10$, with higher A_ψ requiring additional modes.

GROWTH RATES OF SECONDARY INSTABILITIES

Conditions: $Pr = 7, \quad \frac{\kappa_T}{\kappa_S} = 0.01, \quad R_\rho^{-1} = 1.06, \quad k_x = 0.515, \quad \omega = 0.958$

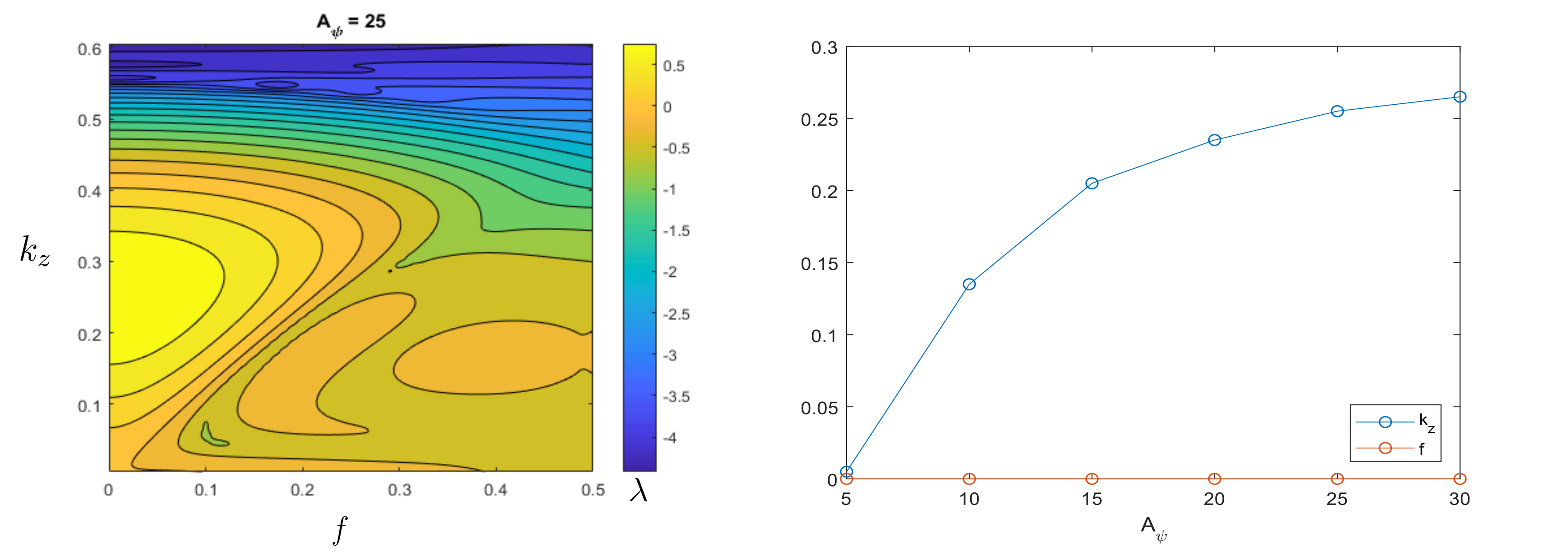


When $A_\psi = 1$ we see an elevator mode ($k_z = 0$). This has the same horizontal scale as the fastest growing mode from linear theory ($k_x = 0.25$, see Fig 1).

When $A_\psi = 3$ the system is unstable to an oscillatory 'intermediate' mode with non-zero f and k_z. This mode has not been observed in previous studies.

When $A_\psi = 6$ the intermediate mode gives way to an oscillatory 'shear' mode with $f = 0$. As A_ψ is increased further, k_z increases in tow.

We investigate the role of heat and salt by studying the stability of oscillatory hydrodynamic (i.e. zero T and S) sinusoidal shear modes. Such systems have been well-studied [5,6]. Note that $1/Pr$ is analagous to a Reynolds number, Re.



The most unstable modes always have $f = 0$, and k_z increases with increasing A_ψ.

In the heat-salt (HS) system, a large A_ψ corresponds to a large Reynolds number, and hence the $f = 0$ mode is likely a shear-driven instability.

In the HS system, at very large Pr (low Re) we see only the intermediate mode, whereas at low Pr (high Re) we see only the shear mode.

CONCLUSIONS AND FUTURE WORK

A large-amplitude basic state forces a shear-driven secondary mode with $f = 0$.

A moderate-amplitude basic state produces a previously undocumented mode with non-zero f and k_z. This is likely a diffusive instability as it is not observed in high-inertia systems, but is prevelent in low-inertia systems.

Secondary instabilities depend on the level of shear -- how does R_ρ^{-1} influence Re?

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